CONTINUITY OF BESSEL POTENTIALS*

BY

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ABSTRACT

It is shown that certain capacities associated with potentials of functions in Lebesgue classes are non-increasing under orthogonal projection of sets. This inequality is then used to discuss continuity of traces of potentials on subspaces of possibly low dimension. The case of principal interest is the Bessel potential.

Introduction

In this paper we will discuss Bessel potentials $g_\alpha * f$ on the Euclidean space $\mathbb{R}^n$, where f is in a Lebesgue space $\mathscr{L}_p$, $1 < p < \infty$. The work will involve the capacity $B_{\alpha,p}$ which has been intensively studied in [1] and [3].

If $M \subset \mathbb{R}^n$ is an affine subspace with $\dim M > n - \alpha p$ then $B_{\alpha,p}(M) > 0$ and it makes sense to speak of the trace of $g_\alpha * f$ on M. In fact the trace on M will be *almost continuous* in the sense of $B_{\alpha,p}$ ($B_{\alpha,p}$-a.c.). On the other hand, if $\dim M \leqq n - \alpha p$ then $B_{\alpha,p}(M) = 0$ and in general no trace will exist on M. However this does not mean that we cannot make interesting statements concerning traces on subspaces of low dimension.

Let $\dim M > n - \alpha p$ and let $M^\perp$ be the largest linear subspace orthogonal to M. We will show, among other things, *$g_\alpha * f$ is a continuous function on the affine subspaces $x + M^\perp$, for x $B_{\alpha,p}$-a.e. in M*. This reveals an interesting duality when coupled with the statement, *$g_\alpha * f$ is $B_{\alpha,p}$-a.c. on the affine subspaces $x + M$, for every x in $M^\perp$*. Thus, if $n - \alpha p < \dim M < \alpha p$ then the potential is $B_{\alpha,p}$-a.c. on $x + M$, for every x in $M^\perp$, and continuous on $x + M$, for x $B_{\alpha,p}$-a.e. in $M^\perp$.

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1. Preliminaries

$\mathbb{R}^n$ will denote the real Euclidean space of points $x = (x_1, \cdots, x_n)$, x_i real, with inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ and norm $|x|$. If A, $B \subset \mathbb{R}^n$ then by $A + B$ we mean the *vector* sum of A and B. We call $M \subset \mathbb{R}^n$ an *affine* subspace, if M is the translate of a linear subspace. We say that $\mathbb{R}^n$ is the *affine* direct sum of affine subspaces M and N, if $\mathbb{R}^n$ is the direct sum of translates of M and N. If M is an affine subspace then $M^\perp$ will be the largest linear subspace orthogonal to M, in the sense that x, $y \in M$ and $z \in M^\perp$ implies $\langle x - y, z \rangle = 0$.

Let $A \subset \mathbb{R}^n$ be locally compact and $\dot{A} = A \cup \{\dot{x}\}$ its one point compactification. By $\mathscr{C}_0(A)$ we mean the Banach space of continuous functions $\phi : A \to \mathbb{R}^1$ with $\lim_{x \to \dot{x}} \phi(x) = 0$, normed by $\max_A |\phi(x)|$. By $\mathscr{C}_c(A)$ we mean the subspace of compact support functions in $\mathscr{C}_0(A)$. $\mathscr{C}_{loc}(A)$ will be the Frechet space of all continuous $\phi : A \to \mathbb{R}^1$, defined by semi-norms $\max_K |\phi(x)|$, $K \subset A$ and compact.

$\mathscr{M}^+ = \mathscr{M}^+(\mathbb{R}^n)$ will be the cone of positive Radon measures on $\mathbb{R}^n$ and will carry the topology of the weak dual of $\mathscr{C}_c(\mathbb{R}^n)$. Each element of $\mathscr{M}^+$ can be identified with the completion of a positive Borel measure which is finite on every compact subset of $\mathbb{R}^n$. If $\mu \in \mathscr{M}^+$ then $\|\mu\|_1$ is the total variation of μ, which may be infinite. Integrals with respect to Lebesgue measure are denoted by $\int \cdots dx$ and notations which don't specifically mention a measure refer to Lebesgue measure. $\mathscr{A}$ will be the σ-algebra of sets which are measurable relative to *every* $\mu \in \mathscr{M}^+$, and if $A \in \mathscr{A}$ then by $\mathscr{M}^+(A)$, we mean the cone of measures in $\mathscr{M}^+$ carried by A.

For A, a Lebesgue measurable set, and $1 < p < \infty$, $\mathscr{L}_p(A)$ will be the usual space of Lebesgue measurable functions $f : A \to [-\infty, +\infty]$ with semi-norm

$$\|f\|_{A;p} = \left(\int_A |f(x)|^p dx \right)^{1/p} < \infty .$$

$\mathscr{L}_p^+(A)$ is the cone of positive functions in $\mathscr{L}_p(A)$. In the case $A = \mathbb{R}^n$ we write $\mathscr{L}_p, \mathscr{L}_p^+$ and $\|f\|_p$.

By a *kernel* k we mean a function $k : \mathbb{R}^n \to [0, +\infty]$ such that k is Borel measurable. Relative to k and $\mathscr{L}_p$ we define a *capacity* $C_{k,p}$ as follows. If $A \subset \mathbb{R}^n$ then

$$C_{k,p}(A) = \inf_f \|f\|_p^p,$$

where $f \in \mathscr{L}_p^+$ and

$$k * f(x) \geqq 1 \text{ for all } x \in A.$$

An f satisfying the above conditions is called a *test function* for $C_{k,p}(A)$, These capacities have been studied extensively in [3], where it is assumed that k is lower semi-continuous. However, this assumption can be dropped in the case of convolutions. Properties of $C_{k,p}$ will be given as needed; however we will state here that $C_{k,p}$ is an outer measure. We may then introduce the familiar concepts of *almost everywhere* ($C_{k,p}$-a.e.) *almost uniform convergence* ($C_{k,p}$-a.u.) and *almost continuous* ($C_{k,p}$-a.c.).

We will be particularly interested in the Bessel kernels g_α, $\alpha > 0$, defined as the inverse Fourier Transform of

$$\hat{g}_\alpha(\xi) = (2\pi)^{-n/2}(1 + |\xi|^2)^{-\alpha/2}.$$

g_α is spherically symmetric, decreasing in $|x|$ and integrable (see section 2[2]). For $C_{g_\alpha,p}$ we write $B_{\alpha,p}$.

Closely associated with $C_{k,p}$ is another capacity, $c_{k,p}$. If $A \in \mathscr{A}$ we define

$$c_{k,p}(A) = \sup_v \| v \|_1,$$

where $v \in \mathscr{M}^+(A)$ and

$$\| k * v \|_{p'} \leqq 1.$$

A v satisfying the above conditions is called a *test measure* for $c_{k,p}(A)$.

2. Monotonicity of energy

If $\mu \in \mathscr{M}^+$, k is a kernel and ϕ is a function satisfying the conditions stated in Lemma 1, then we call

$$\int \phi(k*\mu)(x)dx$$

an energy integral. In this section we study this integral under orthogonal projection of the measure μ.

LEMMA 1. *Let*

$$\phi: [0, +\infty] \to [0, +\infty], \ \phi(0) = 0,$$

be continuous, non-decreasing and let $\phi | [0, +\infty)$ be convex (finite valued). Let

$$k_q: \mathbb{R}^1 \to [0, +\infty) \qquad (q = 1, \cdots, Q)$$

be even functions in $\mathscr{C}_c(\mathbb{R}^1)$ and $k_q | [0, +\infty)$ be non-increasing.

Then if $\eta_q (q = 1, \cdots, Q)$ and η are arbitrary points in $\mathbb{R}^1$,

$$\int \phi\left(\sum_{q=1}^{Q} k_q(x - \eta_q)\right) dx \leqq \int \phi \left(\sum_{q=1}^{Q} k_q(x - \eta)\right) dx.$$

PROOF. We assume that $Q > 1$, for the case $Q = 1$ is trivial. Now suppose that the labelling is such that

$$\eta_1 \leqq \eta_2 \leqq \cdots \leqq \eta_Q.$$

Let $\eta_{Q-1} \leqq \eta' \leqq \eta_Q$; we will show that

$$I = \int \phi\left(\sum_{q=1}^{Q-1} k_q(x-\eta_q) + k_Q(x-\eta_Q)\right) dx \leqq$$

$$II = \int \phi\left(\sum_{q=1}^{Q-1} k_q(x-\eta_q) + k_Q(x-\eta')\right) dx.$$

There is no harm in assuming $\eta' = -\eta_Q$, for we may introduce the new variable $y = x - (\eta' + \eta_Q)/2$. If we rewrite I and II as $\int_{-\infty}^{0} \cdots dx + \int_{0}^{+\infty} \cdots dx$ and make the change of variables x to $-x$ in $\int_0^{+\infty} \cdots dx$, we see that

$$II - I = \int_{-\infty}^{0} \left\{\phi\left(\sum_{q=1}^{Q-1} k_q(x-\eta_q) + k_Q(x+\eta_Q)\right) - \phi\left(\sum_{q=1}^{Q-1} k_q(x-\eta_q) + k_Q(x-\eta_Q)\right)\right\}$$

$$- \left\{\phi\left(\sum_{q=1}^{Q-1} k_q(x+\eta_q) + k_Q(x+\eta_Q)\right) - \phi\left(\sum_{q=1}^{Q-1} k_q(x+\eta_q) + k_Q(x-\eta_Q)\right)\right\} dx.$$

For $x \leqq 0$ we have

$$\sum_{q=1}^{Q-1} k_j(x-\eta_q) \geqq \sum_{q=1}^{Q-1} k_q(x+\eta_q)$$

and

$$k_Q(x+\eta_Q) \geqq k_Q(x-\eta_Q).$$

From the assumptions on ϕ it is clear that $II - I \geqq 0$ or $I \leqq II$. We may then choose $\eta' = \eta_{Q-1}$ and step by step reduce all the η_q to η_1, so that finally

$$I \leqq \int \phi\left(\sum_{q=1}^{Q} k_q(x-\eta_1)\right) dx = \int \phi\left(\sum_{q=1}^{Q} k_q(x-\eta)\right) dx. \qquad \text{Q.E.D}$$

Suppose that $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuous. T can be lifted to a mapping of $\mathcal{M}^+$ into the positive Borel measures on $\mathbb{R}^n$; we call this mapping T also. If $\mu \in \mathcal{M}^+$ and $A \subset \mathbb{R}^n$ is a Borel set then

$$T\mu(A) = \mu(T^{-1}A).$$

In particular we will deal here with the case where M is an affine subspace of $\mathbb{R}^n$ and $T = P_M$ is the orthogonal projection onto M.

THEOREM 1. *Let ϕ satisfy the assumptions of Lemma 1. Let k be a kernel on $\mathbb{R}^n$ such that for $i \geqq d+1$, $0 \leqq d \leqq n-1$, k is even in x_i and non-increasing as a function of x_i for $x_i \geqq 0$. Finally, let M be the affine subspace*

$$x_{d+1} = a_{d+1}, \cdots, x_n = a_n.$$

Then, for all $\mu \in \mathscr{M}^+$,

$$\int \phi(k*\mu)(x)\,dx \leqq \int \phi(k*P_M\mu)(x)\,dx.$$

PROOF. We break the proof up into special cases of increasing generality and pass from one to the next by simple approximations.

Case 1. μ a finite linear combination of Dirac measures; $k \in \mathscr{C}_c(\mathbb{R}^n)$.

We take

$$\mu = \sum_{q=1}^{Q} m_q \delta_{\xi^{(q)}},$$

where $m_q > 0$. We fix the values of $x_1, \cdots, x_{n-1}$ and set

$$k_q(x_n) = m_q k(x_1 - \xi_1^{(q)}, \cdots, x_{n-1} - \xi_{n-1}^{(q)}, x_n) \quad \text{for} \quad q = 1, \cdots, Q.$$

If we set $\eta_q = \xi_n^{(q)}, \eta = a_n$ and let P be the orthogonal projection onto $x_n = a_n$, it is clear from Lemma 1, that

$$\int \phi(k*\mu)(x)\,dx_n \leqq \int \phi(k*P\mu)(x)\,dx_n.$$

Integrating with respect to $x_1, \cdots, x_{n-1}$ we then have

$$\int \phi(k*\mu)(x)\,dx \leqq \int \phi(k*P\mu)(x)\,dx.$$

We now apply this inequality to the successive projections of μ until we reach $P_M\mu$.

Case 2. μ compact support; $k \in \mathscr{C}_c(\mathbb{R}^n)$.

Let $K = \operatorname{supp} \mu$. It is well known that there exists a sequence of measures $\mu_i \in \mathscr{M}^+(K)$ each of which is a finite linear combination of Dirac measures and

$$\| \mu_i \|_1 = \| \mu \|_1, \ \mu_i \to \mu \text{ weakly.}$$

It is clear that

$$\| P_M\mu_i \|_1 = \| P_M\mu \|_1 = \| \mu \|_1 \text{ and } P_M\mu_i \to P_M\mu \text{ weakly.}$$

Therefore

$$k*\mu_i(x) \to k*\mu(x)$$

and

$$k*P_M\mu_i(x) \to k*P_M\mu(x)$$

for all $x \in \mathbb{R}^n$. From Lebesgue's Dominated Convergence Theorem we easily deduce the convergence of the respective energies and hence, from case 1, the inequality.

Case 3. μ compact support; k bounded with compact support.

Given $0 < \varepsilon < 1$, let χ_ε be the indicator function of the cube $|x_1| \leqq \varepsilon, \cdots, |x_n| \leqq \varepsilon$. Set $k_\varepsilon(x) = \varepsilon^{-n}\chi_\varepsilon * k(x)$. The functions $k_\varepsilon \in \mathscr{C}_c(\mathbb{R})$ and satisfy the assumptions of our theorem since k does. The convolutions $k_\varepsilon * \mu$ and $k_\varepsilon * P_M\mu$ are bounded independent of ε and

$$\int |(k_\varepsilon - k)*\mu(x)|\,dx, \int |(k_\varepsilon - k)*P_M\mu(x)|\,dx \to 0$$

as $\varepsilon \to 0$. Since ϕ satisfies a Lipschitz condition on every finite interval, it is clear that the energies of μ and $P_M\mu$ with respect to the k_ε converge to the respective energies with respect to k. The inequality then follows from the preceding case.

Case 4. μ compact support; k general.

By truncating k and cutting it off for large values of $|x|$ we can produce a sequence $\{k_i\}$ satisfying the conditions of Case 3 and

$$k_i(x) \uparrow k(x)$$

for all $x \in \mathbb{R}^n$. Apply Lebesgue's Monotone Convergence Theorem.

Case 5. μ general; k general.

If χ_i is the indicator function of the cube $|x_1| \leqq i, \cdots, |x_n| \leqq i$ define $\mu_i = \chi_i\mu$. Then, from Case 4,

$$\int \phi(k*\mu_i)(x)\,dx \leqq \int \phi(k*P_M\mu_i)(x)\,dx \leqq \int \phi(k*P_M\mu)(x)\,dx.$$

Since, as $i \to \infty$, the energy of μ_i tends to the energy of μ, we are finished.

Q.E.D.

THEOREM 2. *Let ϕ satisfy the conditions of Lemma* 1. *Let k be a kernel on $\mathbb{R}^n$ which is spherically symmetric and non-increasing as $|x|$ increases.*

Then, if M is any affine subspace of $\mathbb{R}^n$ and $\mu \in \mathscr{M}^+$,

$$\int \phi(k*\mu)(x)\,dx \leqq \int \phi(k*P_M\mu)(x)\,dx.$$

PROOF. If $d = \dim M < n$ then define M_0 by

$$x_{d+1} = 0, \cdots, x_n = 0.$$

There is an isometry $R\colon \mathbb{R}^n \to \mathbb{R}^n$ for which $R(M) = M_0$. Hence

$$P_M = R^{-1} P_{M_0} R.$$

Since the energy is invariant under R, the inequality follows from Theorem 1.

Q.E.D.

3. Continuity of potentials

THEOREM 3. *Let k satisfy the assumptions of Theorem 2. If M is an affine subspace of $\mathbb{R}^n$, then*

$$C_{k,p}(P_M A) \leqq C_{k,p}(A),$$

for all $A \subset \mathbb{R}^n$.

PROOF. First we prove it for $A = K$, a compact set. If ν is a test measure for $c_{k,p}(P_M K)$ then it is carried by $P_M K$. By the Hahn-Banach Theorem we can easily produce $\mu \in \mathscr{M}^+(K)$ such that $P_M \mu = \nu$. Hence $\| \mu \|_1 = \| \nu \|_1$ and by Theorem 2, μ is a test measure for $c_{k,p}(K)$. Hence

$$c_{k,p}(P_M K) \leqq c_{k,p}(K).$$

However, for all analytic sets, $c_{k,p} = C_{k,p}^{1/p}$ (see th. 8, [3]).

If A is a countable union of compact sets (i.e. A is a K_σ-set) there is a sequence of compact sets, $K_i \uparrow A$; therefore $P_M K_i \uparrow P_M A$ also. It is known that under these conditions

$$C_{k,p}(K_i) \uparrow C_{k,p}(A) \text{ and } C_{k,p}(P_M K_i) \uparrow C_{k,p}(P_M A);$$

see corollary on p. 265 [3].

To treat the general case let $G \supset A$ be open. Then

$$C_{k,p}(P_M A) \leqq C_{k,p}(P_M G) \leqq C_{k,p}(G).$$

Since $C_{k,p}$ is an outer capacity we are finished; see theorem 1, [3].

Q.E.D.

THEOREM 4. *Let k satisfy the assumptions of Theorem 2 and further let k be locally Lebesgue integrable with $\lim_{|x| \to \infty} k(x) = 0$. Let M be an affine subspace of $\mathbb{R}^n$.*

Then, for $f \in \mathscr{L}_p$ and $\varepsilon > 0$, there exists a closed set $F \subset M$ such that

$$C_{k,p}(M \sim F) < \varepsilon$$

and

$$k{*}f \in \mathscr{C}_0(F + M^{\perp}).$$

Hence,

$$k{*}f \in \mathscr{C}_0(x + M^{\perp}) \qquad C_{k\,p}\text{-a.e. } in\ M.$$

Let $\{f_i\}$ *be a sequence strongly convergent to* f *in* $\mathscr{L}_p$. *Then there is a subsequence* $\{f_{i'}\}$ *such that given* $\varepsilon > 0$, *there exists a closed set* $F \subset M$ *with the property*

$$C_{k,p}(M \sim F) < \varepsilon$$

and

$$k{*}f_{i'} \to k{*}f \ in\ \mathscr{C}_0(F + M^{\perp}).$$

Hence,

$$k{*}f_{i'} \to k{*}f \ in\ \mathscr{C}_0(x + M^{\perp}) \qquad C_{k,p}\text{-a.e. } in\ M.$$

PROOF. There exists a sequence $\{f_i\}$ in $\mathscr{C}_c(\mathbb{R}^n)$ such that $f_i \to f$ strongly in $\mathscr{L}_p$ Clearly

$$k{*}f_i \in \mathscr{C}_0(\mathbb{R}^n).$$

For some subsequence $\{f_{i'}\}$

$$k{*}f_{i'} \to k{*}f \qquad C_{k,p}\text{-a.u.};$$

see theorem 4, [3]. Since $C_{k,p}$ is an outer capacity there is an open set G such that

$$C_{k\,p}(G) < \varepsilon$$

and

$$k{*}f_{i'} \to k * f \text{ uniformly on } \mathbb{R}^n \sim G.$$

We define

$$F = M \sim P_M G.$$

The first part of our theorem then follows from Theorem 3. The second part is proved by practically the same argument. Q.E.D.

THEOREM 5. *Let* M *be an affine subspace of* $\mathbb{R}^n$ *and let* m *be an integer* $0 \leqq m < \alpha$.

For $f \in \mathscr{L}_p$ *and* $\varepsilon > 0$ *there exists a closed set* $F \subset M$ *such that*

$$B_{\alpha-m,p}(M \sim F) < \varepsilon$$

and

$$\left(\frac{\partial}{\partial x}\right)^{\beta}(g_{\alpha} * f) \in \mathscr{C}_0(F + M^{\perp})$$

for all β, $|\beta| \leqq m$. *Hence for such* β,

$$\left(\frac{\partial}{\partial x}\right)^{\beta}(g_{\alpha} * f) \in \mathscr{C}_0(x + M^{\perp}) \ B_{\alpha - m\, p}\text{-a.e. } in\ M.$$

Let $\{f_i\}$ *be a sequence strongly convergent to* f *in* $\mathscr{L}_p$. *Then there is a subsequence* $\{f_{i'}\}$ *such that given* $\varepsilon > 0$, *there exists a closed set* $F \subset M$ *with the property*

$$B_{\alpha - m\, p}(M \sim F) < \varepsilon$$

and

$$\left(\frac{\partial}{\partial x}\right)^{\beta}(g_{\alpha} * f_{i'}) \to \left(\frac{\partial}{\partial x}\right)^{\beta}(g_{\alpha} * f) \text{ in } \mathscr{C}_0(F + M^{\perp})$$

for all β, $|\beta| \leqq m$ *Hence, for such* β,

$$\left(\frac{\partial}{\partial x}\right)^{\beta}(g_{\alpha} * f_{i'}) \to \left(\frac{\partial}{\partial x}\right)^{\beta}(g_{\alpha} * f) \text{ in } \mathscr{C}_0(x + M^{\perp}) \ B_{\alpha - m\, p}\text{-a.e. } in\ M.$$

PROOF. If $|\beta| < \alpha$ then

$$\left(\frac{\partial}{\partial x}\right)(g_{\alpha} * f)(x) = g_{\alpha - |\beta|} * f_{\beta}(x) \quad B_{\alpha - |\beta|, p}\text{-a.e.}$$

where $f_{\beta} \in \mathscr{L}_p$. Furthermore the linear map $f \to f_{\beta}$ is continuous from $\mathscr{L}_p$ into $\mathscr{L}_p$; see section 13, [2]. Also if $|\beta| \leqq m$, then $B_{\alpha - m, p}(A) \leqq B_{\alpha - |\beta|, p}(A)$; this is a simple consequence of the fact that $\int g_{\alpha}(x)\, dx = 1$. Our theorem then follows from Theorems 3 and 4.

Q.E.D.

REMARKS. Of course Theorems 4 and 5 are sometimes devoid of content, for it is possible that $C_{k,p}(M) = 0$ or $B_{\alpha - m, p}(M) = 0$. However in the case of Bessel potentials, if H_d is the d-dimensional Hausdorff measure on $\mathbb{R}^n$,

$$B_{\alpha - m, p}(A) = 0 \text{ implies } H_d(A) = 0$$

for $d > \max(n - (\alpha - m)p, 0)$; see theorem 22, [3]. Therefore if

$$\dim M \geqq d > \max(n - (\alpha - m)p, 0),$$

Theorem 5 is non-trivial and is also non-trivial with H_d-a.e. in place of $B_{\alpha - m, p}$-a.e. Furthermore, if we take $d = \dim M$ and use Sobolev's Inequality for traces on affine subspaces in the manner of theorem 20 of [3], we can substitute

$H_d(M \sim F) < \varepsilon$ for $B_{\alpha-m,p}(M \sim F) < \varepsilon$. For more detailed results on relations between Bessel capacities and Hausdorff measures see [1].

Note the duality implicit in Theorem 5. For $x \in M$, $(\partial/\partial x)^\beta(g_\alpha * f)$ is continuous on $x + M^\perp$ $B_{\alpha-m,p}$-a.e. on M; for $x \in M^\perp$, $(\partial/\partial x)^\beta\ (g_\alpha * f)$ is $B_{\alpha-m\ p}$-a.c. on $x + M$ everywhere on $M^\perp$.

If $u \in W^{\alpha,p}(\mathbb{R}^n)$, the usual Sobolev space, then

$$u(x) = g_\alpha * f(x) \quad \text{a.e.}$$

where $f \in \mathscr{L}_p$ and the map $u \to f$ is an isomorphism of $W^{\alpha,p}(\mathbb{R}^n)$ onto $\mathscr{L}_p$. It follows that if for x we define

$$\tilde{u}(x) = \text{approx.}\lim_{y\to x} u(y)$$

then $\tilde{u}(x)$ is defined $B_{\alpha,p}$-a.e. and

$$\tilde{u}(x) = g_\alpha * f(x) \quad B_{\alpha,p}\text{-a.e.}$$

Further, if $|\beta| < \alpha$ then

$$\left(\frac{\partial}{\partial x}\right)^\beta \tilde{u}(x) = g_{\alpha-|\beta|} * f_\beta(x) \quad B_{\alpha-|\beta|,p}\text{-a.e.}$$

where $f_\beta \in \mathscr{L}_p$ and the map $u \to f_\beta$ is continuous from $W^{\alpha,p}(\mathbb{R}^n)$ into $\mathscr{L}_p$; see section 13, [2]. If $u \in W^{\alpha,p}_{loc}(\Omega)$ and $\phi \in \mathscr{C}^\infty_c(\Omega)$ (Ω an open subset of $\mathbb{R}^n$) then the map $u \to \phi u$ is continuous from $W^{\alpha,p}_{loc}(\Omega)$ into $W^{\alpha,p}(\mathbb{R}^n)$. If we define $\tilde{u}(x)$ as above for $x \in \Omega$ then

$$\left(\frac{\partial}{\partial x}\right)^\beta \phi\tilde{u}(x) = \left(\frac{\partial}{\partial x}\right)^\beta \phi\tilde{u}(x)\ B_{\alpha-m\ p}\text{-a.e.}$$

for $|\beta| < \alpha$.

It is now possible to state a version of Theorem 5 for the Sobolev spaces $W^{\alpha,p}_{loc}(\Omega)$ and $W^{\alpha,p}_0(\Omega)$.

COROLLARY. *Let M be an affine subspace of $\mathbb{R}^n$ and let m be an integer $0 \leqq m < \alpha$.*

For $u \in W^{\alpha,p}_{loc}\ (\Omega)$ there exists a closed set $F \subset P_M\Omega$ such that

$$B_{\alpha-m,p}(P_M\Omega \sim F) < \varepsilon$$

and

$$\left(\frac{\partial}{\partial x}\right)^\beta \tilde{u} \in \mathscr{C}_{loc}((F + M^\perp) \cap \Omega)$$

for all β, $|\beta| \leqq m$. *Hence for such* β,

$$\left(\frac{\partial}{\partial x}\right)^{\beta} \tilde{u} \in \mathscr{C}_{loc}((x + M^{\perp}) \cap \Omega) \ B_{\alpha-m\ p}\text{-a.e. } in \ P_M\Omega.$$

Let $\{u_i\}$ *be a sequence convergent to* u *in* $W^{\alpha,p}_{loc}(\Omega)$. *Then there is a subsequence* $\{u_{i'}\}$ *such that given* $\varepsilon > 0$, *there exists a closed set* $F \subset P_M\Omega$ *with the property*

$$B_{\alpha-m,p}(P_M\Omega \sim F) < \varepsilon$$

and

$$\left(\frac{\partial}{\partial x}\right)^{\beta} \tilde{u}_{i'} \to \left(\frac{\partial}{\partial x}\right)^{\beta} \tilde{u} \ in \ \mathscr{C}_{loc}((F + M^{\perp}) \cap \Omega)$$

for all β, $|\beta| \leqq m$. *Hence, for such* β,

$$\left(\frac{\partial}{\partial x}\right)^{\beta} \tilde{u}_i \to \left(\frac{\partial}{\partial x}\right)^{\beta} \tilde{u} \ in \ \mathscr{C}_{loc}((x + M^{\perp}) \cap \Omega) \ B_{\alpha-m\ p}\text{-a.e. } in \ P_M\Omega.$$

If we replace $W^{\alpha,p}_{loc}(\Omega)$ by $W^{\alpha,p}_0(\Omega)$ and $\mathscr{C}_{loc}$ by $\mathscr{C}_0$ all the above statements remain true.

In the case of $W^{\alpha,p}_0(\Omega)$ the above result tells us how $\tilde{u}$ and its partial derivatives *continuously* assume zero boundary values.

We now wish to study continuity in parallel affine subspaces *not* orthogonal to M. In a sense we have already done this, since such a set of affine subspaces can be indexed by an affine subspace orthogonal to them. Hence the question is simply one of changing the indexing set.

LEMMA 2. *Let* T *be a* $1-1$ *map of* $\mathbb{R}^n$ *onto itself which, together with its inverse* T^{-1}, *satisfies a Lipschitz condition.*

Given ρ, $0 < \rho < \infty$, *if* $A \subset \mathbb{R}^n$ *with* diam $A \leqq \rho$ *then*

$$B_{\alpha,p}(TA) \leqq Q\, B_{\alpha,p}(A);$$

Q *is a constant independent of* A.

PROOF. Let f be a test function for $B_{\alpha\ p}(A)$; then

$$g_\alpha * f(x) \geqq 1 \text{ for all } x \in A.$$

If $x_0 \in A$ is fixed, $A \subset \{|x - x_0| \leqq \rho\} = B(x_0, \rho)$. We may assume

$$\|f\|_p^p \leqq 2B_{\alpha,p}(B(0,\rho)).$$

Since g_α is exponentially decreasing at ∞ (see section 2, [2])

$$\int_{\{|x|\geqq a\rho\}} g_\alpha^{p'}(x)\,dx \leqq 2^{-(p+1)/(p-1)}\, B_{\alpha,p}^{-1/(p-1)}(B(0,\rho)),$$

if $'a'$ is sufficiently large. We fix such a value of $'a'$ and let χ be the indicator function of $B(x_0,(a+1)\rho)$; we set

$$h(x) = 2\chi(x)f(x).$$

Then

$$g_\alpha * h(x) \geqq 1 \text{ for } x \in A,$$

and thus

$$g_\alpha * h(T^{-1}x) \geqq 1 \text{ for } x \in TA.$$

By making a change of variables in the convolution we get

$$\int g_\alpha(L_T^{-1}(x-z))h(T^{-1}z)\; J_T{}^{-1}(z)\,dz \geqq 1 \text{ for } x \in TA,$$

where L_T is the Lipschitz constant for T and $J_{T^{-1}}$ is the Jacobian of T^{-1}. Since $h(T^{-1}z)$ vanishes outside $T(B(x_0,(a+1)\rho))$, for $x \in TA$ we need consider in the above integral, only points z such that $|x-z| \leqq L_T(a+2)\rho$. From the known asymptotic behavior of g_α in the neighborhood of zero (see section 2, [2]) we get

$$g_\alpha(L_{T^{-1}}x) \leqq c\, g_\alpha(x) \text{ for } |x| \leqq L_T(a+2)\rho,$$

where c is a constant independent of x. Therefore $ch(T^{-1}z)\; J_{T^{-1}}(z)$ is a test function for $B_{\alpha,p}(TA)$ and we get

$$B_{\alpha,p}(TA) \leqq (2c)^p(\text{ess sup } J_T)\ (\text{ess sup } J_{T^{-1}})^p\, \|f\|_p^p,$$

from which the inequality follows.

Q.E.D.

If $\mathbb{R}^n$ is the affine direct sum of M and N, we define $P_{M\,N}$ to be the projection of $\mathbb{R}^n$ onto M, parallel to N.

Theorem 6. *Given $0<\rho<\infty$, if $A \subset \mathbb{R}^n$ and* diam $P_{M\,N}A \leqq \rho$ *then*

$$B_{\alpha,p}(P_{M\,N}A) \leqq Q\, B_{\alpha,p}(A),$$

where Q is independent of A.

Proof. The projection $P_N{}^{\perp}$ restricted to M has an affine extension to T, mapping $\mathbb{R}^n$ onto itself. $P_{M,N} = T^{-1}\, P_N^{\perp}$. The result is a consequence of Lemma 2 and Theorem 3.

Q.E.D.

The only essential change we must make in the statement of Theorem 4 in the case M and N *not* orthogonal, is that we must first choose, let us say, a bounded relative open set $G \subset M$, then take $F \subset G$ and write $B_{\alpha-m,p}(G \sim F) < \varepsilon$ in place of $B_{\alpha-m,p}(M \sim F) < \varepsilon$. A similar change must be made in the statement of Theorem 5.

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